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Effective potentials for twisted fields

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Abstract. Minus the density of the effective action, evaluated at the lowest eigenfunction of the (space–time) derivative part of the second (functional) derivative of the classical action, is proposed as a generalised definition of the effective potential, applicable to twisted as well as untwisted sectors of a field theory. The proposal is corroborated by several specific calculations in the twisted sector, namely ϕ^4 theory (real and complex) and wrong-sign-Gordon theory, in an Einstein cylinder, where the exact integrability of the static solutions confirms the effective potential predictions. Both models exhibit a phase transition, which the effective potential locates, and the one-loop quantum shift in the critical radius is computed for the real ϕ^4 model, being a universal result. Topological mass generation at the *classical* level is pointed out, and the exactness of the classical effective potential approximation for complex ϕ^4 is discussed.

1. Introduction

Effective potentials, so useful in studying spontaneous symmetry breaking situations, were brought to the fore by Coleman and Weinberg (1973) (although the method was latent in the work of Jona-Lasinio (1964)) and further developed by several authors (Jackiw 1974, Brown and Duff 1975, Iliopoulos *et al* 1975). The usual definition amounts to

$$V(A) = -(1/\mathrm{VOL})\Gamma(A\phi_0) \tag{1}$$

where Γ is the effective action and $\phi_0(x) = 1$ everywhere.

This definition makes the effective potential rather impotent in the twisted sectors of a field theory, since the only constant twisted field is zero. Thus, if we accept (1) at face value, we can at best hope to calculate $V(0)$ in the twisted sector. For reasons to be described in § 2, we generalise (1) by simply changing ϕ_0 to the lowest eigenfunction of the (space–time) derivative part of the second (functional) derivative of the classical action in the sector under consideration. This is evidently equivalent to the usual definition in non-twisted sectors.

Having decided on such a generalisation, it is rather important to verify that it is useful and, to this end, specific calculations are presented in §§ 3 and 4. Now there is nothing so reassuring as an exact solution, and so we investigate models which in the twisted sector exhibit non-trivial symmetry breaking and critical behaviour at the classical level and are in addition exactly soluble there. Clearly, if our generalisation is to have any credibility at all, it had better reproduce such behaviour properly before we go on to consider closed-loop corrections. We find that it does, and so pass on to calculate the one-loop shift in the critical point, this being universal in a sense to be described in § 4.

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The space–time in which we work will be $T \otimes S^1$ throughout with the metric

$$ds^2 = dt^2 - R^2\, d\theta^2, \qquad \theta \in [0, 2\pi]. \tag{2}$$

This two-dimensional Einstein cylinder is non-simply connected and therefore admits non-trivial twisted and automorphic field configurations (Isham 1978a, b, Banach and Dowker 1979, Banach 1980a), which in the case of real scalar fields just means antiperiodicity in θ, and for complex scalars, a periodicity condition like

$$\phi(t, 2\pi) = e^{2\pi i\alpha}\phi(t, 0) \tag{3}$$

where α is arbitrary and specifies a distinct sector of the theory. The models we discuss are two-phase ϕ^4 theory, two-phase $|\phi|^4$ theory and wrong-sign-Gordon theory.

2. The generalised effective potential

We begin by writing down some familiar formulae, more to fix notation than anything else; the reader who does not know what they mean will find adequate explanation in the effective potential references cited above or in Abers and Lee (1973); ϕ denotes some multiplet of scalar fields for definiteness.

$$Z(J) = e^{(i/\hbar)W(J)} = \int [d\phi] \exp\{(i/\hbar)(S(\phi) + J\phi)\}, \tag{4}$$

$$\Phi = \delta W/\delta J, \tag{5}$$

$$\Gamma(\Phi) = W(J) - J\Phi, \tag{6}$$

$$\delta\Gamma/\delta\Phi = -J. \tag{7}$$

$\Gamma(\Phi)$ is the effective action. It has a functional Taylor series expansion which displays the $1PI$ proper vertices of the theory governed by the action S:

$$\Gamma(\Phi) = \Gamma(\Phi_G) + \sum_{n=2}^{\infty} \frac{1}{n!} \frac{\delta^n\Gamma}{\delta\Phi_1 \ldots \delta\Phi_n}\bigg|_{\Phi=\Phi_G} (\Phi_1 - \Phi_G) \ldots (\Phi_n - \Phi_G) \tag{8}$$

where Φ_G is the ground state of the theory given by

$$\delta\Gamma/\delta\Phi|_{\Phi=\Phi_G} = 0. \tag{9}$$

Alternatively, we may change variables and go to momentum space. By momentum space in this context we mean little more than the spectrum of the derivative terms in $\delta^2S/\delta\phi^2$. As a general rule they will constitute a self-adjoint operator in a Hilbert space of functions belonging to a given sector of the field theory (we don't mean to be too precise about this here), which will have a spectral decomposition, and we can write heuristically

$$\Phi = \sum_n a_n f_n \tag{10}$$

where the f_n are eigenfunctions and the a_n are some coefficients.

Equation (8) then becomes

$$\Gamma(\Phi) = \Gamma(\Phi') + \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i_1 \ldots i_n} \frac{\delta^n\Gamma}{\delta a_{i_1} \ldots \delta a_{i_n}}\bigg|_{\Phi=\Phi'} a_{i_1} \ldots a_{i_n} \tag{11}$$

where

$$\Phi = \Phi' + \sum_n a_n f_n \tag{12}$$

and we have allowed for the possibility that Φ' is not the ground state.

Now suppose that the ground state of the theory is $\Phi_G = 0$; then, setting $\Phi = A\Phi'$ where Φ' is *any* function, we will find that $\Gamma(\Phi) = \Gamma(A\Phi')$ will have a minimum at $A = 0$:

$$\mathrm{d}\Gamma(A\Phi')/\mathrm{d}A|_{A=0} = 0. \tag{13}$$

Thus *any* ray through the origin in function space will locate the correct vacuum state. However, if $\Phi_G \neq 0$ is the ground state, only the ray $\Phi = A\Phi_G$ will locate the true vacuum, i.e.

$$\mathrm{d}\Gamma(A\Phi_G)/\mathrm{d}A|_{A=1} = 0, \tag{14}$$

and not only will (14) hold at this point in function space, but so will (9). Unfortunately, as a rule, we don't know Φ_G.

The customary escape from this impasse in untwisted sectors (the only ones where it is applicable) is to use symmetry arguments. It is stated that one is only interested in cases where the vacuum expectation value of ϕ is translationally invariant, which immediately cuts down the problem to an essentially one-dimensional one—to search for Φ_G within the ray of constant fields. Since, in general, the lowest states of a quantum theory carry the simplest representations of any symmetry groups that may be available, and a constant carries the trivial representation of the translation group, this turns out to be a very good choice.

Can one then mimic the above behaviour in more general contexts where constants are suppressed for topological reasons? Clearly we want a family of functions to try, for which the symmetry behaviour is as simple as possible. In addition, we would like as little contribution as possible from the derivative terms in the action, since these only help to raise the energy. The origin of momentum space (as defined above) seems to be a reasonable place to look; accordingly we define our generalised effective potential as

$$V(A) = -(1/\text{VOL})\Gamma(A\phi_0) \tag{15}$$

where VOL is the volume of space–time and ϕ_0 is an eigenfunction belonging to the lowest eigenvalue of the derivative terms of $\delta^2 S/\delta\phi^2$ which we henceforth call the lowest momentum state.

A disclaimer is immediately called for. We do *not* pretend that some multiple of ϕ_0 will be the ground state of the system. In fact, if $V(A)$ has a minimum for some non-zero value of A, A_0 say, then $\delta\Gamma/\delta\Phi|_{\Phi=A_0\phi_0}$ will in general not be zero, i.e. there will be contributions at the true ground state from the other eigenvalues of the kinetic terms. However, particularly in symmetry restoration contexts, it will be shown that there is a regime, near the critical point, in which the higher terms in (11) (with Φ' set to $A\phi_0$) can be neglected, and that the first term (itself in the one-loop approximation) gives the leading behaviour, i.e. that (15) contains some useful physics.

In this connection, we can ask what difference expansion about an approximate rather than true ground state makes to the effective action. The answer is — surprisingly little. Thus let us suppose that ϕ_c is a stationary point of the classical action,

$$\delta S/\delta\phi|_{\phi=\phi_c} + J = 0, \tag{16}$$

making ϕ_c a functional of J. In the region of small J (since we want to turn off the external source eventually) let ε, given by

$$\phi_0 = \phi_c + \varepsilon, \tag{17}$$

be small where ϕ_0 is some fixed field about which we want to expand $Z(J)$ near $J=0$. We can perform the one-loop integration as usual — it just has an extra linear piece in the exponential — finding

$$W(J) = S(\phi_0) + J\phi_0 - \tfrac{1}{2}\varepsilon S''(\phi_0)\varepsilon + \tfrac{1}{2}S'''(\phi_0)\varepsilon^3 \ldots + \tfrac{1}{2}\mathrm{i}\hbar\, \mathrm{Tr}\ln S''(\phi_0). \tag{18}$$

Remembering that ϕ_0 is independent of J, we find

$$\Phi = \phi_0 - \varepsilon - \varepsilon^2 S'''(\phi_0)[1 - \tfrac{3}{2}(S''(\phi_0))^{-1}] \ldots \tag{19}$$

whence

$$\Gamma(\Phi) = S(\Phi) + \tfrac{1}{2}\mathrm{i}\hbar\, \mathrm{Tr}\,[\ln S''(\Phi) + (S''(\Phi))^{-1} S'''(\Phi)\varepsilon] \tag{20}$$

correct to $O(\hbar\varepsilon)$ and to second order in ε. Thus there is little change introduced by a small shift in the point about which we integrate. This is because the error is effectively one in the external source, which then cancels in leading orders in the Legendre transformation.

In the following sections of this paper we will only calculate the usual one-loop expression for the effective action (i.e. the first two terms of (20)). This will imply the presence of some external source J if Φ is not the correct extremum of Γ. At the critical point, this will disappear, but even away from the critical point the implied J will be negligible provided $\varepsilon^2 \sim \hbar$.

3. ϕ^4 theory

We consider, in the twisted sector, the model given by the action

$$S = \frac{R}{2}\int \mathrm{d}t \int_0^{2\pi} \mathrm{d}\theta \left(\dot{\phi}^2 - \frac{1}{R^2}\phi'^2 - \frac{\lambda^2}{2}(\phi^2 - a^2)^2\right). \tag{21}$$

This model was investigated in depth at the classical level by Avis and Isham (1978) and we merely restate the most relevant facts here, referring to the original paper for further details.

For static solutions, the equation of motion is exactly soluble and the stable ground state solution is given as follows;

$$\phi_G = 0 \qquad \text{if } R \leq (2\lambda a)^{-1}, \tag{22a}$$

$$\phi_G = (a^2 - w)^{1/2}\, \mathrm{sn}\left[\lambda R\left(\frac{a^2+w}{2}\right)^{1/2}(\theta + \theta_0); \frac{a^2 - w}{a^2 + w}\right] \qquad \text{if } R \geq (2\lambda a)^{-1}, \tag{22b}$$

where sn is a Jacobi elliptic function (Abramowitz and Stegun 1964) and w is given by

$$\pi\lambda R\left(\frac{a^2+w}{2}\right)^{1/2} = \int_0^{\pi/2} \frac{\mathrm{d}u}{\{1 - [(a^2-w)/(a^2+w)]\sin^2 u\}^{1/2}}. \tag{22c}$$

The phase transition occurs (classically) at $R = (2\lambda a)^{-1} = R_c^0$, at which point we see that w, which is a constant of integration, is a^2. A notable point is that ϕ_G is uniformly continuous in R.

Now we treat the same system using the effective potential method suggested above. The lowest momentum state of the action (21) is the lowest mode of

$$\frac{\partial^2}{\partial t^2}-\frac{1}{R^2}\frac{\partial^2}{\partial \theta^2} \tag{23}$$

which, under the circumstances, is $A\cos[\frac{1}{2}(\theta+\theta_0)]$. To evaluate the effective potential in tree approximation we simply insert this into (21), evaluate the result, divide by $2\pi R\int \mathrm{d}t$ and change the overall sign. The answer is

$$V^0(A)=\tfrac{1}{4}[\lambda^2a^4+A^2(1/4R^2-\lambda^2a^2)+\tfrac{3}{8}\lambda^2A^4] \tag{24}$$

and we see the classical phase transition displayed in standard Landau fashion, which is to say that for $R<R_c^0$ the minimum of V^0 is at zero, while for $R>R_c^0$ the minimum is at some non-zero field. For $R<R_c^0$ the solution is clearly exact, i.e. the minimum of V^0 identifies the true ground state $\phi_G=0$, while for $R>R_c^0$ the minimum gives an approximation to the ground state — an increasingly good one the nearer one goes to R_c^0, as can most simply be seen by the limiting behaviour of $\mathrm{sn}(\theta; m)$ as $m\to 0$, in which limit it becomes $\sin\theta$. Clearly, the phase transition is being dominated by long-wavelength phenomena which we have simply highlighted.

At this point, it is appropriate to comment on our choice of trial field ϕ_0. Had we chosen a field with some admixture of shorter-wavelength components, the implied phase transition would have been at a different point; the coefficient of R^{-2} in (24) would have been a number exceeding $\frac{1}{4}$ *unless* the trial field just happened to reduce to $A\cos[\frac{1}{2}(\theta+\theta_0)]$ at $R=R_c^0$. In this respect, the choice of 'lowest momentum state' as trial field has served us remarkably well in view of the rather vague reasons used in making it.

Heartened by our success at tree approximation and encouraged by the limiting exactness of our method near the phase transition, we proceed to tempt providence at the one-loop level.

The Euclideanised second derivative of the action (21) evaluated at $A\cos(\frac{1}{2}\theta)$ is

$$\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}=-R\left(-\frac{\partial^2}{\partial t^2}-\frac{1}{R^2}\frac{\partial^2}{\partial\theta^2}-\lambda^2a^2+3\lambda^2A^2\cos^2(\tfrac{1}{2}\theta)\right)\delta(x,y) \tag{25}$$

and the one-loop correction to the effective potential is given by

$$V^1=-\frac{1}{\mathrm{VOL}}\frac{\hbar}{2}\operatorname{Tr}\ln\left(\frac{\delta^2 S}{\delta\phi^2}\right) \tag{26}$$

which we define (given that it is divergent) by zeta function regularisation

$$V^1=\frac{-1}{\mathrm{VOL}}\frac{\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}s}\operatorname{Tr}\left(\frac{\delta^2S}{\delta\phi^2}\right)^s\bigg|_{s=0}=\frac{-1}{\mathrm{VOL}}\frac{\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left(\sum_i (L\lambda_i)^s\right)\bigg|_{s=0} \tag{27}$$

where the λ_i are the eigenvalues of S'' and L is a length scale introduced for renormalisation purposes.

Now (25) may be related to a Mathieu equation for which the eigenvalues are known as series in the coefficient of the trigonometric term. This means that we can find the spectrum of (25) as a set of power series in A^2 (some of the details are described in the Appendix) if we separate variables and assume $\mathrm{e}^{i\omega t}$ behaviour for the time variable. The expression we have to evaluate then becomes (absorbing a minus sign into the definition of L)

$$V^1(A)=\frac{-1}{2\pi RT}\frac{\hbar}{2}\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left(\frac{L}{R}\right)^s\left(\frac{T}{2\pi}\right)\int \mathrm{d}\omega \right.$$
$$\times[\tfrac{1}{4}(1+3R^2\lambda^2A^2-\tfrac{9}{8}R^4\lambda^4A^4)+R^2\omega^2+R^2\lambda^2(\tfrac{3}{2}A^2-a^2)]^s$$
$$+[\tfrac{1}{4}(1-3R^2\lambda^2A^2-\tfrac{9}{8}R^4\lambda^4A^4)+R^2\omega^2+R^2\lambda^2(\tfrac{3}{2}A^2-a^2)]^s$$
$$+2\sum_{m=1}^{\infty}\left[\frac{1}{4}\left((2m+1)^2+\frac{9R^4\lambda^4A^4}{2[(2m+1)^2-1]}\right)+R^2\omega^2\right.\tag{28}$$
$$\left.\left.+R^2\lambda^2(\tfrac{3}{2}A^2-a^2)\right]^s\right\}_{s=0}$$

to the required order in A^2. Equation (28) is an analytic function of A^2 at $A^2=0$, so after performing the integral over ω which is elementary, we can expand in A^2†. The coefficients in the series (to $\mathrm{O}(A^4)$) involve summations like

$$\sum_{m=1}^{\infty}\left(\frac{(2m+1)^2}{4}-R^2\lambda^2a^2\right)^z \qquad \text{and} \qquad \sum_{m=1}^{\infty}\frac{1}{[(2m+1)^2-1]}\left(\frac{(2m+1)^2}{4}-R^2\lambda^2a^2\right)^z \tag{29}$$

which can be expressed as summations of Riemann zeta functions by expanding in powers of $R^2\lambda^2a^2$—a strategy which exposes their singularity structures to the naked eye. The poles in (29) at the relevant points are cancelled by a $\Gamma(-s)^{-1}$ factor from the ω integration and the overall expression for $V^1(A)$ is finite.

The final expression for $V^1(A)$ consists of two parts. The first comes from the first two terms in (28), which themselves are a consequence of the special form, to this order, of the lowest two eigenvalues of the Mathieu operator (see the Appendix),

$$V_0^1(A)=(\hbar/4\pi R^2)\{-(1-4R^2\lambda^2a^2)^{1/2}-3A^2R^2\lambda^2(1-4R^2\lambda^2a^2)^{-1/2}$$
$$+\tfrac{1}{8}A^4R^4\lambda^4[45(1-4R^2\lambda^2a^2)^{-3/2}+\tfrac{9}{2}(1-4R^2\lambda^2a^2)^{-1/2}]\},\tag{30}$$

and the second comes from the summation in (28), which accounts for all the other eigenvalues,

$$V_\infty^1(A)=(\hbar/2\pi R^2)\Big[[\tfrac{11}{24}-R^2\lambda^2a^2(2K-\tfrac{1}{2})-\tfrac{1}{2}Q(2,-1)]$$
$$+A^2R^2\lambda^2[3K-3Q(1,1)]-A^4R^4\lambda^4\Big(9Q(0,3)$$
$$+\frac{9}{8}\sqrt{\pi}\sum_{r,l=0}^{\infty}\frac{(-4R^2\lambda^2a^2)^l}{\Gamma(l+1)\Gamma(-\frac{1}{2}-l)}[(1-2^{-[3+2(l+r)]})\zeta[3+2(l+r)]-1]\Big)\Big]\tag{31}$$

where

$$K=\tfrac{1}{2}-\tfrac{1}{4}\gamma+\tfrac{1}{4}\ln 2\approx 0.529\tag{32}$$

and

$$Q(p,q)=\frac{\sqrt{\pi}}{2}\sum_{l=p}^{\infty}\frac{(-4R^2\lambda^2a^2)^l}{\Gamma(l+1)\Gamma(1-l-\frac{1}{2}q)}[(1-2^{-(2l+q)})\zeta(2l+q)-1].\tag{33}$$

† Reversing the order of these steps gives a slightly more compact form for the final expression which, however, has the disadvantage of obscuring somewhat the singularity structure evident in (30).

Actually this is not quite the whole story. There is in addition the contribution

$$V_R^1(A) = (\hbar\lambda^2/8\pi)(\tfrac{3}{2}A^2 - a^2)\ln(L/R). \tag{34}$$

The terms in (34) are a mass term and a vacuum energy (Casimir) term which we renormalise away by adding the counterterms

$$\tfrac{1}{2}\delta m^2\phi^2 + \delta c \tag{35}$$

(where δm^2 and δc are both $O(\hbar)$) to the classical Lagrangian density in (21), and the final answer is the one we would have obtained by setting L to R in (28). This procedure is equivalent to the usual practice of consistently setting the mass etc to given quantities at some chosen momentum scale.

Were it not for the factor of 3 in (34), we could have achieved both renormalisations by a shift $a^2 \to a^2 + \delta a^2$ in the classical Lagrangian, as is clear by examining how this replacement affects the classical effective potential (24). However, not only is the factor of 3 really there (it is the same factor of 3 that appears in (25)) but setting $a^2 = 0$ obviates the need for the Casimir term while leaving the mass term unaffected. So this proposal, though tempting, is inappropriate.

Our final expression for the effective potential thus becomes (all constants now being renormalised ones)

$$V(A) = V^0(A) + V_0^1(A) + V_\infty^1(A). \tag{36}$$

This expression, though frankly a mess when written out in full detail, has several features worthy of note.

(i) To the given order in A (i.e. fourth) it is exact. This is pleasing in view of the fact that we have exact expressions for neither the eigenvalues of $\delta^2 S/\delta\phi^2$ nor the subsequent zeta function. Everything is due to the fact that these quantities have power series expressions which are regular at $A = 0$. Other approaches to the eigenvalue problem (e.g. the asymptotic method of Dikii (1961)) would not have given the correct form without much further effort, if at all.

(ii) At no stage in the derivation of (36) were we ever explicitly confronted with the problem of zero-frequency translation modes. As has been pointed out above, the fact that we are *not* expanding around the ground state implies the presence of a non-zero space-dependent J in the system, which is here responsible for breaking translation invariance.

(iii) Our expression becomes complex for $R > R_c^0$ and A small enough. It is clear from (28) and (30) that this behaviour sets in when the lowest eigenvalues of $\delta^2 S/\delta\phi^2$ at $\phi = 0$ become negative, other eigenvalues following suit at larger values of R. Dolan and Jackiw (1974), in their discussion of symmetry restoration at finite temperature, also find the one-loop effective potential to be complex for small ϕ. They attribute this to the breakdown of the one-loop approximation and confirm it by a large-N limit calculation. However, our case, unlike theirs, has a regime ($R < R_c^0$) where the one-loop expression is unexceptionable and so presumably reliable, giving a sensible value for the critical radius (see below), so it may be that some other effect is responsible. For instance, complex effective actions are usually associated with decay situations and particle production, so it may be that there is simply no real J that gives rise to the relevant Φ for $R > R_c^0$ and that such a Φ is consequently unstable.

This phenomenon should not be confused with the consequences of zero-frequency modes, two of which happen to appear at $4R^2\lambda^2 a^2 = 1$, $A = 0$ but nowhere else.

(iv) Despite superficial appearances to the contrary, the sign of A^4 in (31) is positive, since the first terms of the summations in its coefficient are negative and each term is typically an order of magnitude down on its predecessor. We are therefore not imperilled by instability against unconstrained growth of Φ.

(v) We can examine limiting cases of the theory. As $\lambda \to 0$, $A \to 0$ the theory becomes that of a massless twisted scalar field in $T \otimes S^1$ and the limiting value of the effective potential, namely $-\hbar/48\pi R^2$, is the correct one to give agreement with the total energy calculated for this system by Dowker and Banach (1978) and Isham (1978a). If we let $R \to \infty$, keeping $R\lambda$ finite, the whole expression vanishes as we would expect. If we set a^2 to 0, we have a massless twisted ϕ^4 theory, and we see clearly topological mass generation not only at the one-loop level (cf Ford and Yoshimura 1979) but at the *classical* level as well. This is clear from the $\lambda^2 \to 0$ limit of the classical potential (24) which is $A^2/16R^2$. Thus, in the absence of *any* potential in the Lagrangian, we get a minimum quadratic contribution to the effective action, which is therefore interpretable as a mass term (any other non-zero twisted field would give an even bigger contribution). The topological mass to one-loop level is then (a^2 is still zero)

$$m_{\mathrm{T}}^2 = 1/8R^2 - (3\hbar\lambda^2/\pi)(\tfrac{1}{2} + K). \tag{37}$$

(vi) The universal covering space of $T \otimes S^1$ is two-dimensional Minkowski space. In a recent publication (Banach 1980b) it was suggested that the renormalisation procedures in multiply connected spaces should be the same as in their universal covers. Thus we ought to compare our results with, say, kink renormalisation in $\mathbb{R}^2$ (e.g. Rajaraman 1975), and indeed we find that a vacuum and mass (but no coupling constant) renormalisation are required for the kink. It is more difficult actually to compare the counterterms themselves, since different regularisation procedures were used in the two calculations and in addition the fundamental group involved is infinite, a case (strictly speaking) not covered in the paper referred to.

The effective potential (36) enables us to compute the one-loop correction to the critical radius. The critical radius is that value of R which makes the coefficient of A^2 in (36) vanish. If we neglect the small corrections due to the term $Q(1, 1)$ in (31), we obtain a cubic equation for R^2 which can be solved exactly. The leading behaviour of its real root yields

$$R_{\mathrm{c}} = \frac{1}{2\lambda a}\left[1 - \frac{\hbar K}{a^2\pi} - 4.06\left(\frac{\hbar^2}{a^4\pi^2}\right)^{1/3} \ldots\right]. \tag{38}$$

Note that R_{c}^0 has been decreased. This must be a pure quantum effect (quite apart from the factors of $\hbar$) since, as we have remarked above, classical effects can only increase R_{c}^0. It is also extremely fortunate in view of the complex nature of $V^1(A)$ above R_{c}^0. Further remarks on (38) are postponed to the discussion.

Finally in this section, we will consider a complex ϕ^4 model given by the action

$$S = \frac{R}{2}\int \mathrm{d}t \int_0^{2\pi} \mathrm{d}\theta\left(|\dot{\phi}|^2 - \frac{1}{R^2}|\phi'|^2 - \frac{\lambda^2}{2}(|\phi|^2 - a^2)^2\right) \tag{39}$$

for which there is a one-parameter family of automorphic sectors labelled by a parameter α and specified by the periodicity condition (3), a consequence of the O(2) symmetry of the theory. They interpolate smoothly between the standard ($\alpha = 0$) and 'maximally twisted' ($\alpha = \frac{1}{2}$) cases. The equation of motion following from (39) is

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{R^2}\frac{\partial^2 \phi}{\partial \theta^2} + \lambda^2(|\phi|^2 - a^2)\phi = 0, \tag{40}$$

which for static solutions is exactly soluble with the ground state (in the sector given by (3))

$$\phi_G = 0 \qquad \text{if } R \leqslant \alpha/\lambda a, \tag{41a}$$

$$\phi_G = (a^2 - \alpha^2/\lambda^2 R^2)^{1/2}\, e^{i\alpha\theta} \qquad \text{if } R \geqslant \alpha/\lambda a. \tag{41b}$$

Actually, a detailed stability analysis of (41) has not been carried out by the author, but the similarity to the real scalar field is strong enough for us to be confident of the results. The classical critical radius is thus

$$R_c^0(\alpha) = \alpha/\lambda a. \tag{42}$$

The effective potential approach offers a surprise in this model. The lowest momentum state is $A\, e^{i\alpha\theta}$, which is of the form of (41b), so that the effective potential approach will be *exact* here. Inserting this into (39) and dividing by minus the volume, we have

$$V^0(A) = \tfrac{1}{2}[\tfrac{1}{2}\lambda^2 a^4 + A^2(\alpha^2/R^2 - \lambda^2 a^2) + \tfrac{1}{2}\lambda^2 A^4]. \tag{43}$$

V^0 thus predicts the phase transition correctly and, further, the amplitude of the optimal ground state, given by $\mathrm{d}V^0/\mathrm{d}A = 0$, is also correctly given to agree with (41b).

While the one-loop corrections are unavailable (we would have to solve a coupled eigenvalue problem), the classical approximation is already very useful. The lowest momentum approximation to the ground state is exact, just as it is for untwisted fields in familiar cases, and the smooth interpolation we find from untwisted to twisted sectors gives us added confidence that our proposal for a twisted effective potential is physically credible.

4. Wrong-sign-Gordon theory

Wrong-sign-Gordon theory is given by the action

$$S = R \int \mathrm{d}t \int_0^{2\pi} \mathrm{d}\theta \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2R^2}\phi'^2 - \frac{m^2}{\lambda^2}[1 + \cos(\lambda\phi)] \right) \tag{44}$$

or, equivalently, the equation of motion

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{1}{R^2}\frac{\partial^2 \phi}{\partial \theta^2} - \frac{m^2}{\lambda}\sin(\lambda\phi) = 0 \tag{45}$$

which differs from conventional sine-Gordon theory in the sign of the sine. Had we retained the original sign, there would have been no sign of the phase transition it is our design to display.

For static solutions, (45) readily yields a first integral

$$\phi'^2 = C + (2R^2 m^2/\lambda^2)[1 + \cos(\lambda\phi)]. \tag{46}$$

The substitution $\xi^2 = 1 - \cos(\lambda\phi)$ then gives us

$$\pm\sqrt{2}Rm(\theta+\theta_0) = \int_0^{[1-\cos(\lambda\phi)]^{1/2}} \frac{2\mathrm{d}\xi}{[(2-\xi^2)(2+C\lambda^2/2R^2m^2-\xi^2)]^{1/2}}. \tag{47}$$

Bearing in mind that $C<0$ (since ϕ' must have a zero somewhere in $[0, 2\pi]$), we can evaluate (47) with the help of Abramowitz and Stegun (1964) and find

$$\phi = (1/\lambda)\cos^{-1}[1-2(1-w)\,\mathrm{sn}^2(Rm(\theta+\theta_0);\,1-w)] \tag{48}$$

where

$$w = -C\lambda^2/4R^2m^2. \tag{49}$$

The antiperiodicity requirement fixes w to be the largest solution satisfying $0 \leqslant w \leqslant 1$ of

$$\pi Rm = \int_0^{\pi/2} \frac{\mathrm{d}u}{[1-(1-w)\sin^2 u]^{1/2}}. \tag{50}$$

As $w \to 1$ the solution (48) tends to zero, which happens when

$$R = R_c^0 = 1/2m. \tag{51}$$

This then is the classical critical point below which the stable ground state is the zero field. A detailed stability analysis of the solution has again not been carried out by the author, but the behaviour is so similar to that of the ϕ^4 model that he once more feels confident. Added confirmation of this comes via the $R\to\infty$, Rm = constant limit of the theory where $w \to 0$, and the solution (48) reduces to

$$\begin{aligned}\phi_\infty &= (1/\lambda)\cos^{-1}\{1-2\tanh^2[Rm(\theta+\theta_0)]\}\\ &= (1/\lambda)[\![4\tan^{-1}\{\exp[Rm(\theta+\theta_0)]\}-\pi]\!]\end{aligned} \tag{52}$$

which is the usual sine-Gordon soliton allowing for the unconventional sign, as one would expect.

We can now see how the effective potential method fares in this case. The lowest momentum state is again $A\cos[\frac{1}{2}(\theta+\theta_0)]$. We put this into (44), make the sign and volume adjustments and find

$$V^0(A) = A^2/16R^2 + (m^2/\lambda^2)[1+J_0(\lambda A)]. \tag{53}$$

The convexity of A^2 competes with the local concavity of J_0 near the origin, and we again find that the phase transition occurs at precisely the correct point, namely R_c^0. However, unlike the ϕ^4 model, as we increase R further, other minima of the potential appear besides the double-well ones as more of the oscillations of J_0 survive the debilitating effect of the A^2 term. These minima presumably correspond to familiar multi-soliton solutions of sine-Gordon theory.

We may remark at this stage that had we retained the conventional sign in (44), the sign of the J_0 in (53) would have been negative, destroying the competition and with it the phase transition.

To consider the one-loop corrections, we write down the required second derivative:

$$\frac{\delta^2 S}{\delta\phi(x)\,\delta\phi(y)} = -R\left(-\frac{\partial^2}{\partial t^2} - \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} - m^2 \cos[\lambda A \cos(\tfrac{1}{2}\theta)]\right)\delta(x, y)$$

$$= -R\left[-\frac{\partial^2}{\partial t^2} - \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} - m^2\left(1 - \frac{\lambda^2 A^2}{2}\cos^2(\tfrac{1}{2}\theta) + \frac{\lambda^4 A^4}{4!}\cos^4(\tfrac{1}{2}\theta)\ldots\right)\right]\delta(x, y)$$

$$= -R\left\{-\frac{\partial^2}{\partial t^2} - \frac{1}{R^2}\frac{\partial^2}{\partial\theta^2} - m^2\left[\left(1 - \frac{\lambda^2 A^2}{4} + \frac{3\lambda^4 A^4}{192}\ldots\right)\right.\right.$$

$$\left.\left. - \cos\theta\left(\frac{\lambda^2 A^2}{4} - \frac{\lambda^4 A^4}{48}\ldots\right) + \cos 2\theta\left(\frac{\lambda^4 A^4}{192}\ldots\right)\ldots\right]\right\}\delta(x, y). \tag{54}$$

Taking it that $\partial^2/\partial t^2$ merely contributes an ω^2 term, we are left with a case of Hill's equation. An examination of the general formulae for the solution of this eigenvalue problem (McLachlan 1947, p 135) yields the following information. The lowest order to which the $\cos 2\theta$ term contributes is A^6, so it may be ignored if we only work to A^4. We are now left with a Mathieu operator. Rather than go through a similar analysis to that for the ϕ^4 model, producing an equally opaque final result, we note the following. The A^4 terms in the constant and $\cos\theta$ terms in (54) themselves only contribute at A^4 and above. Therefore, to find the critical radius for which we only need the A^2 term, we can drop these corrections. This procedure is now equivalent to dropping the $\cos^4(\frac{1}{2}\theta)$ term in (54) in its entirety, and we see that what is left has entirely the same structure as (25). To obtain the effective potential, correct to A^2, we therefore merely have to make the substitutions

$$\lambda^2 \to \lambda^2 m^2/6, \tag{55a}$$

$$a^2 \to 6/\lambda^2. \tag{55b}$$

The quantum corrected critical radius is thus

$$R_c = \frac{1}{2m}\left[1 - \frac{\hbar K \lambda^2}{6\pi} - 1.23\left(\frac{\hbar^2\lambda^4}{\pi^2}\right)^{1/3}\ldots\right] \tag{56}$$

and we can now see how the critical radius is a universal result. Given any classical potential $V(\phi)$, satisfying $\partial^2 V/\partial\phi^2|_{\phi=0} < 0$, in the twisted sector of $T \otimes S^1$ with the standard kinetic terms, we will find that the second derivative of the action at $\phi = A\cos(\frac{1}{2}\theta)$ will have an expansion like (54). While the numerical coefficients may be different, the power-counting arguments above will continue to hold, and we will find that we can drop all but the constant and $\cos^2(\frac{1}{2}\theta)$ terms. Simple replacements like (55) will then give the correct critical radius. This completes our discussion of wrong-sign-Gordon theory.

5. Discussion

In the preceding sections we have made out what is, to the author's mind, a strong case for a generalised effective potential. Most of the corroboration, as we have seen, took place at the classical level where there were non-trivial classical solutions to check against. Features have emerged which are very reminiscent of condensed matter theory, namely fractional powers of coupling constants (and, by rescaling, of $\hbar$) in expressions for the critical radius (more commonly, the inverse critical temperature) and 'universality' of parameters characterising the phase transition. Several further points do however suggest themselves.

(1) Does the method still give good answers when there is non-trivial time dependence in the lowest momentum state which would, among other things, produce a difference between the classical energy and the classical action?

(2) Does the method still give the best answers when there is a non-trivial quadratic part in the Lagrangian? In the examples above, there was no essential difference (other than a shift in eigenvalue) between taking the lowest momentum state and the lowest eigenfunction of the quadratic part of the Lagrangian. Would the latter prescription yield better results if there was (say) a non-trivial conformal coupling, $\xi R\phi^2$, in the action?

(3) How do we systematically improve upon our lowest-order guess for the ground state Φ_G in the absence of translation-invariance type arguments? For instance, would the inclusion of more degrees of freedom (i.e. higher momentum contributions) significantly improve the one-loop values (as opposed to the classical values) or not?

(4) What is the true significance of the complex nature of $V(A)$ above the classical phase transition? It is clear that it signals some sort of perturbative inadequacy or instability in the theory, but the details of the process responsible are far from evident and the issue deserves further elucidation. Perhaps the best prospect for this lies with the wrong-sign-Gordon model, where we would hope for a more complete quantum solution by analogy with sine-Gordon theory in two-dimensional Minkowski space. This possibility will be explored in a subsequent paper.

Despite these points, it should nevertheless be the case that our determination of the critical radius is exact, since at the critical radius the ground state is exactly zero and thus small errors in the classical field above the critical radius should be insignificant to the order to which we have calculated.

The way is now open for investigating other models using the methods presented here. We have been lucky in our present investigation, insofar as non-trivial twisted field effects typically happen at one order less in the coupling constant than for standard fields, e.g. symmetry restoration, occurrence of fractional powers of coupling constants in R_c, topological mass generation. This has enabled us to build up confidence by comparison with exact results.

We now ought to investigate further the complex ϕ^4 model to one-loop, where first indications suggest that the corrections diverge in the untwisted limit (I am indebted to R Critchley for this information). We ought also to tackle models that have been treated by other means in the literature, especially those which have been found to be unstable at zero field, namely the interaction of an untwisted and a twisted scalar field in $S^1 \otimes \mathbb{R}^3$ (Toms 1980, Ford 1980b) and QED in $S^1 \otimes \mathbb{R}^3$ (Ford 1980a, b). All of these are multicomponent systems and, to determine the one-loop corrections, coupled eigenvalue problems have to be solved. If, as here, all that is required is the first few terms, then no serious difficulty is encountered if one employs the methods used for the Mathieu equation (see e.g. McLachlan 1947).

In conclusion, the method advocated here shows promise of elucidating twisted sectors away from zero field. Although not exact in general, it has shown itself to be very good in the critical region and so to be a reliable guide to at least the qualitative behaviour of a theory.

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Appendix

For the sake of completeness, we collect together some formulae pertaining to the Mathieu equation, to enable the reader better to understand formula (28). Mathieu's equation is

$$-\mathrm{d}^2y/\mathrm{d}z^2+2q\cos(2z)y=\mu y. \tag{A1}$$

It is assumed that y and μ have power series expansions in q, and equating powers of q then yields the explicit forms (see McLachlan 1947). The periodicity requirement makes (A1) into an eigenvalue problem, the standard solutions having period 2π and satisfying

$$y(z+n\pi)=(\pm)^n y(z). \tag{A2}$$

It is the ones belonging to $(-)^n$ in (A2) that we need, as the re-expression of (25) in terms of $2u=\theta$ rapidly shows. The required eigenvalues are

$$\begin{aligned}\mu &= 1-q-\tfrac{1}{8}q^2\ldots,\\ \mu &= 1+q-\tfrac{1}{8}q^2\ldots\end{aligned} \tag{A3}$$

and

$$\mu=(2m+1)^2+\frac{q^2}{2[(2m+1)^2-1]}\cdots \qquad \text{doubly degenerate, } m\geqslant 1. \tag{A4}$$

Comparison of (A1) and (25) eventually leads to (28).

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